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BOUNDEDLY COMPLETE BASIC SEQUENCES, c_0-SUBSPACES, AND INJECTIONS OF BANACH SPACES

BY

V.P. FONF*

Department of Mathematics, Ben-Gurion University of the Negev
P.O.B. 653, Beer Sheva 84105, Israel

ABSTRACT

We study the connection between topological properties of subsets of a given Banach space and their images under linear, continuous one-to-one mappings on the one hand and the existence in a given Banach space of either a boundedly complete basic sequence (BCBS) or an isomorphic copy of c_o (c_o-subspace) on the other hand. We present criteria for the existence of a BCBS. They are deduced from new characterisations of G_δ-embeddings which we also present. We obtain a necessary and sufficient condition for separability of a dual Banach space in terms of saturation by BCBS. Criteria for the existence in a Banach space of a c_o-subspace are also presented. We describe the class of separable Banach spaces which contains either a BCBS or a c_o-subspace.

Introduction

The series of striking counterexamples that were constructed recently by Gowers and Maurey [11] and Gowers [12] completely dispersed all hopes of a simple linear-topological structure of infinite-dimensional Banach spaces. The most delicate conjecture:

Every infinite-dimensional Banach space contains either a boundedly complete basic sequence (BCBS) or a subspace isomorphic to the space c_o (c_o-subspace)

has been disproved also.

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The main purpose of this paper is to describe the class $\mathcal{K}$ of separable Banach spaces that contains either a BCBS or a c_o-subspace. It turned out that the separable Banach space E belonging to the class $\mathcal{K}$ is equivalent to the existence of an injection T: $E \to X$ (by injection we mean a linear continuous one-to-one map into some Banach space X with unbounded inverse T^{-1}) with special properties. We will be interested in properties of the injections T that are connected with Borelian type of images TA (of some subsets $A \subset E$) both in the whole space X and in the image TE. Let us note that topological properties of the set TA in the image TE coincide with those of the set A in the X-topology on the space E (by X-topology on the space E we mean the topology that is generated by sets $T^{-1}(G)$ where G is an open subset of the space X; for example, an X-ball in the space E is the set $T^{-1}(B)$ where B is some ball in the space X). To distinguish the X-topology and the original norm-topology on the space E, we will denote the latter by E-topology.

The above-mentioned characterization of the class $\mathcal{K}$ is contained in the following theorem.

THEOREM 7: *Let E be a separable Banach space. Then the following assertions are equivalent:*

(1) $E \in \mathcal{K}$.

(2) *There exist an injection T: $E \to X$ (into some Banach space X) and a non-empty bounded open subset $A \subset E$ such that either the image TA is of the type $G_{\delta\sigma}$ in the space X or the image TA is of the type G_δ in the image TE.*

(3) *There exist an injection T: $E \to Y$ (into some Banach space Y) and a non-empty bounded open subset $B \subset E$ such that either the image TB is of the type F_σ in the space Y or the image $T(cB)$ is of the type F_σ in the image TE.*

We will examine this theorem by two approaches. The treatment from the BCBS is contained in part 1. The main tool here will be the notion of a G_δ-embedding that was introduced and studied by Bourgain and Rosenthal [1]. We recall that an injection T: $E \to X$ of the Banach space E into the Banach space X is a G_δ-embedding iff the image TA of every closed, bounded and separable subset $A \subset E$ is a G_δ-set in the space X. Important properties of G_δ-embeddings were obtained by Ghoussoub and Maurey [9,10]. The papers of Edgar and Wheeler [2] and Rosenthal [18] discuss closely related topics. We will use some ideas from these papers as well as from previous papers of the author [4–7].

The main results of part 1 are Theorem 1, which gives the characterization of G_δ-embeddings, and Theorem 2, which gives the most general criterion for the existence of BCBS in a given Banach space (in terms of injections).

The approach to Theorem 7 from a c_o-subspace is contained in part 2. We will use here a notion of a set of super-first category that was introduced by the author in his paper [8]. The main result of part 2 is Theorem 5, which characterizes Banach spaces that contain a c_o-subspace.

The short part 3 combines results of parts 1 and 2.

We will assume that all Banach spaces considered are real and infinite dimensional (unless specified otherwise). We use standard Banach space theory notations as can be found in [16], to which we refer the reader for unexplained terminology. By $U(E)$ $(S(E))$ we denote the unit ball (unit sphere) of the linear normed space E. In part 1, $T\colon E \to X$ denotes an injection into the Banach space X.

1. Injections and BCBS

Let $\epsilon\colon \Sigma \to R_+$ be a map from the set Σ of all ordered finite subsets of the unit sphere $S(E)$ into the set of positive numbers R_+. We will say that the map ϵ is a T-regulator of boundedly complete basic sequences (briefly: T-RBCBS) iff every sequence $x_n \subset S(E)$ possessing the following property:

$$(*) \qquad \text{For every } n = 1, 2, \ldots, \quad \|Tx_{n+1}\| \leq \epsilon(\{x_i\}_1^n)$$

is BCBS.

It is obvious that every X-null sequence $\{x_n\} \subset S(E)$ (i.e. a null-sequence in the X-topology) has a subsequence possessing the property $(*)$. So existence of T-RBCBS implies: every X-null sequence from a unit sphere has a subsequence which is BCBS.

We will use the following

Proposition 1: [17] *Let E be a separable Banach space. The following assertions are equivalent:*

(1) *T^*X^* is a norming linear manifold.*

(2) *T^{-1} belongs to the first Baire class.*

(3) *$X - \mathrm{cl}\, U(E)$ is a bounded subset of the space E.*

Remark 1: If T^*X^* is 1-norming then the unit ball $U(E)$ is X-closed.

The following theorem is the main result of this part.

THEOREM 1: *Let T: $E \to X$ be an injection of the separable Banach space E into the Banach space X such that T^{-1} is a map of the first Baire class. The following assertions are equivalent:*
(1) There exists the bounded subset $D \subset E$ such that its image TD is a G_δ-set in X and $E - \text{cl}\, D$ contains some E-ball.
(2) There exists a T-RBCBS.
(3) The map T is a G_δ-embedding.

Proof: Since T^{-1} is a map of the first Baire class the linear manifold T^*X^* is norming. Without loss of generality we can assume that T^*X^* is 1-norming (and therefore (Remark 1) the unit ball $U(E)$ is X-closed) and $\|T\| \leq 1$.

(1) $\Rightarrow$ (2). Let $G = TD = \bigcap_1^\infty G_n$, where each set G_n is an open subset of the space X. Without loss of generality we can assume that $E - \text{cl}\, D \supset 2U(E)$ and $0 \in D$. Denoting $D_n = T^{-1}(G_n)$ we have $D = \bigcap_1^\infty D_n$. Since T^{-1} is a map of the first Baire class, there exists a sequence $\{g_n\}$ of continuous mappings g_n: $TE \to E$ such that, for all $x \in E$, $\lim g_n(Tx) = x$. Put

$$\omega(g, x, \delta) = \sup\{\alpha\colon \|x - y\| \leq \alpha \Rightarrow \|g(x) - g(y)\| \leq \delta\}.$$

Let $\{\delta_n\}$ be a sequence of positive numbers such that

$$\sum_1^\infty \delta_n < 1/8, \quad \prod_1^\infty (1 + \delta_n)/(1 - 2\delta_n) \leq 2.$$

We begin the construction of the map ϵ with $n = 1$. Let $x_1 \in S(E)$; then there exists an element $\bar{x}_1$ possessing the properties:
(a) $\|x_1 - \bar{x}_1\| < \delta_1$.
(b) There exists a δ_1-net $\{t_i^1 \bar{x}_1\}_{i=1}^{l_1}$ in the segment $[-\bar{x}_1, \bar{x}_1]$ which is contained in the set D_1, i.e.

$$\{t_i^1 \bar{x}_1\}_1^{l_1} \subset [-\bar{x}_1, \bar{x}_1] \cap D_1$$

(recall that the open set D_1 is dense in the set $2U(E)$). Put

$$d(x_1) = d_X(\{Tt_i^1 \bar{x}_1\}_1^{l_1}, \partial G_1).$$

It is evident that there exists a number $r(x_1)$ such that, for all $i = 1, 2, \dots, l_1$ and $m \geq r(x_1)$,

$$\|g_m(T(t_i^1\bar{x}_1)) - t_i^1\bar{x}_1\| < \delta_1.$$

Denote

$$t(x_1) = \min\{\omega(g_{r(x_1)}, t_i^1\bar{x}_1, \delta_1): i = 1, 2, \dots, l_1\}.$$

Since the linear manifold T^*X^* is 1-norming, there exists a linear functional $h_1 \in X^*$ such that $\|T^*h_1\| = 1$ and $(T^*h_1)(x_1) \geq (1 - \delta_1)$. Finally, put

$$\epsilon(\{x_1\}) = \min\{d(x_1), t(x_1)\}\delta_1^2(8\|h_1\|\|T\|)^{-1}.$$

Now we will define the map ϵ on two-element subsets $\{x_1, x_2\} \subset S(E)$. Since the open set D_2 is dense in the ball $2U(E)$ there exists a vector $\bar{x}_2$ possessing the properties:

(a) $\|x_2 - \bar{x}_2\| < \epsilon(\{x_1\})$.

(b) There exists a δ_2-net $\{t_i^2\bar{x}_2\}_1^{l_2}$ in the segment $[-\bar{x}_2, \bar{x}_2]$ which is contained in the set D_2 and the set

$$A(x_1, x_2) = \{t_i^1\bar{x}_1 + t_j^2\bar{x}_2 \in U(E): 1 \leq i \leq l_1, 1 \leq j \leq l_2\}$$

is also contained in D_2. Put

$$d(x_1, x_2) = d_X(TA(x_1, x_2), \partial G_2).$$

Let $r(x_1, x_2)$ be such a number that for all $m \geq r(x_1, x_2)$ and for all $x \in A(x_1, x_2)$, $\|g_m(Tx) - x\| < \delta_2$. Denote

$$t(x_1, x_2) = \min\{\omega(g_{r(x_1,x_2)}, x, \delta_2): x \in A(x_1, x_2)\}.$$

Since the linear manifold T^*X^* is 1-norming, there exists a finite subset $\{h_i\}_1^{m_2} \subset X^*$ such that $\{T^*h_i\}_1^{m_2} \subset S(E^*)$ and the set

$$\{(T^*h_i)|_{[x_1,x_2]}\}_1^{m_2}$$

is a δ_2-net in the ball $U([x_1, x_2]^*)$. We denote by $[x, y]$ the linear span of x and y. Finally, put

$$\epsilon(\{x_1, x_2\}) = \min_{1 \leq k \leq 2}\{d(\{x_i\}_1^k), t(\{x_i\}_1^k)\}\delta_2^2(8\|T\|\max\ \{\|h_i\|: i = 1, \dots, m_2\})^{-1}.$$

Continuing in such a way we construct the map $\epsilon\colon \Sigma \to R_+$. Now let the sequence $\{y_i\} \subset S(E)$ satisfy $\|Ty_{n+1}\| \le \epsilon(\{y_i\}_1^n)$ for all $n = 1, 2, \dots$. We have to prove that $\{y_i\}$ is a BCBS. At first we will check that $\{y_i\}$ is a basic sequence. Here we act standardly:

$$\begin{aligned}\Big\|\sum_1^n a_i y_i\Big\| &\le (1-\delta_n)^{-1}\max\left\{\left|T^*h_i\left(\sum_1^n a_i y_i\right)\right|\colon 1 \le i \le m_n\right\}\\ &= (1-\delta_n)^{-1}\max\left\{\left|T^*h_i\left(\sum_1^{n+1} a_i y_i\right) - T^*h_i(a_{n+1}y_{n+1})\right|\colon 1 \le i \le m_n\right\}\\ &\le (1-\delta_n)^{-1}\Big(\max\left\{\|T^*h_i\|\Big\|\sum_1^{n+1} a_i y_i\Big\|\colon 1 \le i \le m_n\right\}\\ &\quad + |a_{n+1}|\max\{|h_i(Ty_{n+1})|\colon 1 \le i \le m_n\}\Big),\end{aligned}$$

but

$$|a_{n+1}| \le \Big\|\sum_1^{n+1} a_i y_i\Big\| + \Big\|\sum_1^n a_i y_i\Big\|$$

and

$$|h_i(Ty_{n+1})| \le \|h_i\|\|Ty_{n+1}\| < \delta_n.$$

Therefore

$$\Big\|\sum_1^n a_i y_i\Big\| \le (1-\delta_n)^{-1}\left(\Big\|\sum_1^{n+1} a_i y_i\Big\| + \delta_n\left(\Big\|\sum_1^{n+1} a_i y_i\Big\| + \Big\|\sum_1^n a_i y_i\Big\|\right)\right)$$

and we have finally

$$\Big\|\sum_1^n a_i y_i\Big\| \le (1+\delta_n)(1-2\delta_n)^{-1}\Big\|\sum_1^{n+1} a_i y_i\Big\|.$$

Using the inequality

$$\prod_1^\infty (1+\delta_n)(1-2\delta_n)^{-1} \le 2$$

we get that $\{y_i\}$ is a basic sequence with basis constant less than 2. By the Krein–Milman–Rutman stability theorem (recall that $\Sigma_1^\infty \delta_n < 1/8$) it is easily verified that $\{\bar{y}_i\}$ ($\bar{y}_i$ has the same sense with respect to y_i as $\bar{x}_i$ has with respect to x_i) is a basic sequence which is equivalent to $\{y_i\}$. Direct verification shows that the basis constant of $\{\bar{y}_i\}$ is less than 5. So it will be enough to check that $\{\bar{y}_i\}$ is a BCBS. We begin with the weaker property. Namely, let

$$\sup\left\{\Big\|\sum_1^n t_{i_k}^k \bar{y}_k\Big\| : n = 1, 2, \dots\right\} < 1/10. \tag{1}$$

Let us check that the series $\Sigma t_{i_k}^k \bar{y}_k$ converges. We have

$$\|T\bar{y}_k\| \le \|T(y_k - \bar{y}_k)\| + \|Ty_k\| \le (1 + \|T\|)\epsilon(\{y_i\}_1^{k-1}) < 2\delta_{k-1}$$

and from (1) we get $|t_{i_k}^k| \le 1, k = 1, 2, \dots$. So the series $\Sigma t_{i_k} T\bar{y}_k$ converges. Put $x_o = \Sigma t_{i_k}^k T\bar{y}_k$. Let us show that for all $n = 1, 2, \dots, x_o \in G_n$. We have

$$\begin{aligned}\Big\|\sum_{n+1}^\infty t_{i_k}^k T\bar{y}_k\Big\| &\le \sum_{n+1}^\infty \|T\bar{y}_k\| \le (1 + \|T\|)\sum_{n+1}^\infty \epsilon(\{y_i\}_1^{k-1}) \\ &\le 2\sum_{n+1}^\infty \delta_{k-1} d(\{y_i\}_1^{k-1}) \le 2d(\{y_i\}_1^n)\sum_1^\infty \delta_k \le d(\{y_i\}_1^n)/4.\end{aligned}$$

Since $\Sigma_1^n t_{i_k}^k \bar{y}_k \in U(E)$, and by the definition of $d(\{y_i\}_1^n)$, we get that $x_o \in G_n$, $n = 1, 2, \dots$. So $x_o \in \bigcap_1^\infty G_n \subset TE$ and therefore $\lim g_n(x_o)$ exists. Denoting $r_n = r(\{y_i\}_1^n)$ we have:

$$\begin{aligned}&\Big\|g_{r_n}(x_o) - \sum_1^n t_{i_k}^k \bar{y}_k\Big\| \\ &= \Big\|g_{r_n}\left(\sum_1^\infty t_{i_k}^k T\bar{y}_k\right) - g_{r_n}\left(\sum_1^n t_{i_k}^k T\bar{y}_k\right) + g_{r_n}\left(\sum_1^n t_{i_k}^k T\bar{y}_k\right) - \sum_1^n t_{i_k}^k \bar{y}_k\Big\| \\ &\le \Big\|g_{r_n}\left(\sum_1^n t_{i_k}^k T\bar{y}_k + \sum_{n+1}^\infty t_{i_k}^k T\bar{y}_k\right) - g_{r_n}\left(\sum_1^n t_{i_k}^k T\bar{y}_k\right)\Big\| \\ &\quad + \Big\|g_{r_n}\left(\sum_1^n t_{i_k}^k T\bar{y}_k\right) - \sum_1^n t_{i_k}^k \bar{y}_k\Big\|.\end{aligned}$$

But

$$\begin{aligned}\Big\|\sum_{n+1}^\infty t_{i_k}^k T\bar{y}_k\Big\| &\le \sum_{n+1}^\infty \|T\bar{y}_k\| \le 2\sum_{n+1}^\infty \epsilon(\{y_i\}_1^{k-1}) \\ &\le 2t(\{y_i\}_1^n)\sum_1^\infty \delta_k \le 1/4\omega\left(g_{r_n}, \sum_1^n t_{i_k}^k T\bar{y}_k, \delta_n\right)\end{aligned}$$

and therefore

$$\|g_{r_n}(x_o) - \sum_1^n t_{i_k}^k \bar{y}_k\| < 2\delta_n.$$

Now it is easily seen that the series $\Sigma t_{i_k}^k \bar{y}_k$ converges. Finally let $\sup\{\|\Sigma_1^n p_k \bar{y}_k\| \colon n = 1, 2, \ldots\} \leq 1/20$. Then $\sup|p_k| < 1/2$ and hence, for each integer k, there exists a number $t_{i_k}^k$ such that $|p_k - t_{i_k}^k| \leq \delta_k$. Thus

$$\sup \|\sum_1^n t_{i_k}^k \bar{y}_k\| \leq 1/4$$

and, as proved above, the series $\Sigma t_{i_k}^k \bar{y}_k$ (and hence also $\Sigma p_k \bar{y}_k$) converges. This completes the proof of implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). It is evident that for every sequence $\{x_i\} \subset S(E)$ the series $\Sigma \epsilon(\{x_i\}_1^n)$ converges. To prove (2) $\Rightarrow$ (3) assume the contrary, i.e. T is not G_δ-embedding. Then from [9] there exist a number $\delta > 0$ and a sequence $\{y_i\} \subset U(E)$ such that $\|y_i - y_j\| \geq \delta, i \neq j$, but the sequence $\{Ty_i\}$ is dense in itself. Put $x_1 = y_1, e_o = x_1/\|x_1\|$ and choose an element y_{n_2} such that

$$\|Ty_1 - Ty_{n_2}\| < \delta\epsilon(\{e_o\}).$$

Put $x_2 = y_{n_2}, e_{11} = (x_1 - x_2)/\|x_1 - x_2\|$. Then

$$\|Tx_1 - Tx_2\| < \delta\epsilon(\{e_o\}), \quad \|Te_{11}\| < \epsilon(\{e_o\}). \tag{2}$$

Using density of the sequence $\{Ty_i\}$ in itself again we choose an element y_{n_3} such that $\|Tx_1 - Ty_{n_3}\| < \delta\epsilon(\{e_o, e_{11}\})$. Put $x_3 = y_{n_3}$ and $e_{21} = (x_1 - x_3)/\|x_1 - x_3\|$. Then

$$\|Tx_1 - Tx_3\| < \delta\epsilon(\{e_o, e_{11}\}), \quad \|Te_{21}\| < \epsilon(\{e_o, e_{11}\}). \tag{3}$$

Let y_{n_4} be such an element that $\|Tx_2 - Ty_{n_4}\| < \delta\epsilon(\{e_o, e_{11}, e_{21}\})$. Denoting $x_4 = y_{n_4}, e_{22} = (x_2 - x_4)/\|x_2 - x_4\|$ we obtain

$$\|Tx_2 - Tx_4\| < \delta\epsilon(\{e_o, e_{11}, e_{21}\}), \quad \|Te_{22}\| < \epsilon(\{e_o, e_{11}, e_{21}\}). \tag{4}$$

The sequence $\{e_o, e_{ij}\}$ will be constructed in this way. By condition (2) of the theorem, $\{e_o, e_{ij}\}$ is BCBS. Let $\{z_i\}$ be the sequence $\{e_o, e_{ij}\}$ that is numerated by one index and put $Y = [z_i]_1^\infty$. We shall show that $T|_Y$ is a G_δ-embedding. Let us introduce the new norm

$$|||y||| = \sup \Big\| \sum_1^n a_i z_i \Big\|, \quad y = \sum a_i z_i$$

in Y which is equivalent to the original one. Denote by $V = \{y \in Y\colon |||y||| \leq 1\}$ the unit ball in the new norm. We will show that the image TV is closed in the space X. Put

$$u_m = \sum a_i^m z_i \in V, \quad m = 1, 2, \ldots, \quad \lim Tu_m = v.$$

Without loss of generality we can assume that for all $i = 1, 2, \ldots$ there exists $\lim a_i^m = a_i$. It is evident that $\sup \|\Sigma_1^n a_i z_i\| \leq 1$. Since $\{z_i\}$ is a BCBS series $\Sigma a_i z_i$ converges to some element u. Hence $u = \Sigma a_i z_i \in V$. Fix ϵ and choose a number n such that $\Sigma_{n+1}^\infty \epsilon(\{z_i\}_1^{k-1}) < \epsilon/16$. There exists a number m such that, for all $i = 1, \ldots, n$, $|a_i^m - a_i| < \epsilon/(4n\|T\|)$. We have:

$$\|Tu_m - Tu\| \leq \Big\| \sum_1^n (a_i^m - a_i) T z_i \Big\| + \Big\| \sum_{n+1}^\infty (a_i^m - a_i) T z_i \Big\| < \epsilon.$$

Hence $\lim Tu_m = Tu, u \in V, v = Tu$ and therefore $v \in TV$. Thus $T|_Y$ is a semi-embedding and, since Y is separable, T is a G_δ-embedding. But the image $\{Tx_i\}$ of δ-separated sequence $\{x_i\} \subset U(Y)$ (remember that $\{x_i\} \subset \{y_i\}$) is dense in itself (see (2), (3), (4)). This is impossible [9]. So implication (2) $\Rightarrow$ (3) is proved.

Implication (3) $\Rightarrow$ (1) is evident. The proof of the theorem is completed. ∎

Remark 2: If an injection $T\colon E \to X$ of a separable Banach space E into a Banach space X is a G_δ-embedding, then T^{-1} belongs to the first Baire class (see [1, 6]).

Remark 3: We do not use the separability of the space E in the proof of implication (1) $\Rightarrow$ (2).

The following theorem gives the most general (in terms of injections) criterion for the existence of BCBS in a given separable Banach space without the assumption that T^{-1} belongs to the first Baire class.

THEOREM 2: *Let $T: E \to X$ be an injection of a separable Banach space E into a Banach space X. If there exists a bounded subset $D \subset E$ which is dense in some ball of the space E and whose image TD is a G_δ-set in the space X, then $E \supset BCBS$.*

Proof: If the inverse mapping T^{-1} belongs to the first Baire class, then by Theorem 1, $E \supset BCBS$. Suppose that T^{-1} does not belong to the first Baire class. Hence, by Proposition 1, the set $X - \operatorname{cl} U(E)$ is unbounded. Denote $V = \operatorname{cl} TU(E)$ and, by Z, the Banach space lin$V$ with the set V as the unit ball. Let $T_1: E \to Z$, $T_2: Z \to X$ be natural embeddings. Since the set $X - \operatorname{cl} U(E)$ is unbounded, it follows that T_1 is not an isomorphic embedding. We will show that the inverse mapping T_1^{-1} belongs to the first Baire class, or equivalently (by Proposition 1): $Z - \operatorname{cl} U(E)$ is bounded in the space E. Without loss of generality we may assume that $E - \operatorname{cl} D \supset U(E)$. By the conditions of the theorem, there exists a sequence $\{G_n\}$ of open subsets of the space X such that $TD = \bigcap G_n$. Denote $D_n = T^{-1}(G_n)$ and let $y_o \in Z - \operatorname{cl} U(E), y = 1/2y_o$. It is evident that y belongs to the algebraic interior of the set $Z - \operatorname{cl} U(E)$. Hence there exists a number $\gamma > 0$ such that $y + \gamma U(E) \subset Z - \operatorname{cl} U(E)$. We will consider two cases.
(1) For every $\delta \in (0,\gamma), D \cap (y + \delta U(E)) \neq \emptyset$.
Then $y \in E - \operatorname{cl} D$ and therefore

$$\sup\{\|y_o\|\colon y_o \in Z - \operatorname{cl} U(E)\} \leq 2 \sup\{\|z\|\colon z \in E - \operatorname{cl} D\}.$$

(2) There exists $\delta \in (0,\gamma), D \cap (y + \delta U(E)) = \emptyset$.

Then by $D = \bigcap D_k$ we have $(y + \delta U(E)) \subset \bigcup cD_k$. By the Baire category theorem, there exist a number m and an E-ball $W \subset y + \delta U(E)$ such that $cD_m \supset W$. Since the set cD_m is X-closed, we obtain

$$X - \operatorname{cl} W \subset cD_m. \tag{5}$$

On the other hand,

$$W \subset y + \gamma U(E) \subset Z - \operatorname{cl} U(E) \subset Z - \operatorname{cl} D$$

hence $(X - \operatorname{cl} W) \cap D \neq \emptyset$ (the set $X - \operatorname{cl} W$ is a Z-ball in the space E and therefore it is a Z-neighborhood for every point from the algebraic interior of the set W). We have obtained the contradiction to (5).

Thus the set $Z - \mathrm{cl}\,U(E)$ is bounded and, by Proposition 1, inverse mapping T_1^{-1} belongs to the first Baire class. But $T_1 D = \cap T_2^{-1}(G_n)$ and every set $T_2^{-1}(G_n)$ is open in the space Z. So by Theorem 1 (applied to the injection $T_1\colon E \to Z$), $E \supset BCBS$. The proof is completed. ∎

The following corollary is a consequence of Theorem 2 and the Baire category theorem.

COROLLARY 1: *Let $T\colon E \to X$ be an injection of a separable Banach space E into a Banach space X such that the image $TU(E)$ of the unit ball $U(E)$ of the space E is a $G_{\delta\sigma}$-set in the space X. Then $E \supset BCBS$.*

Remark 4: The restriction on the Borel type of the image $TU(E)$ cannot be weakened (if we want to say anything about the space E) because for every injection $T\colon E \to X$ with the inverse from the first Baire class, for any separable Banach space E, the image $TU(E)$ is an $F_{\sigma\delta}$-set in X [7].

Now we pass to the characterization of separability of the dual space E^* in terms of the saturation by BCBS.

Let $W\colon \Sigma \to \mathcal{B}$ be a map of the set of all ordered finite subsets of the unit sphere $S(E^*)$ into the set $\mathcal{B}$ of all w^*-neighborhoods of zero in the unit ball $U(E^*)$. We will say that the map W is a w^*-regulator of boundedly complete basic sequences (briefly: w^*-RBCBS) if and only if every sequence $\{f_n\} \subset S(E^*)$ possessing the property:

$$(**) \qquad \text{For all } n = 1, 2, \ldots, \quad f_{n+1} \in W(\{f_i\}_1^n)$$

is BCBS.

It is obvious that every w^*-null sequence $\{f_n\} \subset S(E^*)$ has a subsequence possessing the property $(**)$. Let us note that according to the well-known result of Johnson and Rosenthal [13] every w^*-null sequence from the unit sphere of a separable dual space has a BCB subsequence. Thus the following theorem strengthens the result of Johnson and Rosenthal bringing it to a necessary and sufficient condition.

THEOREM 3: *Let E be a separable Banach space. The following assertions are equivalent:*

(1) *The dual space E^* is separable.*

(2) *There exists a w^*-RBCBS.*

Proof: (1) $\Rightarrow$ (2). Let $A\colon l_2 \to E$ be some compact operator from the separable Hilbert space l_2 into the space E with dense range. Denote $T = A^*\colon E^* \to l_2$. Then T is a semi-embedding (the image of the unit ball is closed) and, by separability of the space E^*, it follows that T is a G_δ-embedding [1]. Now assertion (2) follows from Theorem 1 ((3) $\Rightarrow$ (2)) since w^*-topology on the unit ball $U(E^*)$ coincides with the l_2-topology.

(2) $\Rightarrow$ (1). Let T be the operator introduced above. With the help of Theorem 1 ((2) $\Rightarrow$ (3)) it is easily verified that T is a G_δ-embedding (recall that the notion of G_δ-embedding is separable defined). We will prove that every w^*-compact subset $K \subset U(E^*)$ is w^*-huskable (i.e. for every $\epsilon > 0$ and every w^*-open subset D possessing the property $D \cap K \neq \emptyset$ there exists a w^*-open subset $D_1 \subset D$ such that $D_1 \cap K \neq \emptyset$ and $\operatorname{diam}(D_1 \cap K) < \epsilon$). Let $\{f_i\}$ be a countable w^*-dense subset of the set K (the space E is separable) and D be an w^*-open subset of the space E^* such that $D \cap K \neq \emptyset$. Denote $K_1 = \|.\| - \mathrm{cl}\,\{f_i\}_1^\infty$ and $K_2 = \|.\| - \mathrm{cl}\,(K_1 \cap D)$. So K_2 is a non-empty (by w^*-density of the set K_1 in the set K and by $K \cap D \neq \emptyset$) separable bounded closed subset as well as K_1. Since T is a G_δ-embedding there exists [9] a point of continuity of the map $T^{-1}|_{TK_2}$. Hence by compactness of the operator T there exists a point $g \in K_2$ and w^*-open neighborhood D_1 of g such that $D_1 \cap K_2 \neq \emptyset$ and $\operatorname{diam}(D_1 \cap K_2) < \epsilon$. As the set $K_1 \cap D$ is dense in the set K_2, there exists an element $g_1 \in (K_1 \cap D) \cap D_1$. Denoting $D_2 = D \cap D_1$ we get $D_2 \cap K_1 \neq \emptyset$. Since $K_1 \cap D_2 = (K_1 \cap D) \cap D_1 \subset K_2$ and $D_2 \subset D$, then $K_1 \cap D_2 \subset K_2 \cap D_1$ and by $\operatorname{diam}(K_2 \cap D_1) < \epsilon$ we have $\operatorname{diam}(K_1 \cap D_2) < \epsilon$. It is clear that $\operatorname{diam}(w^* - \mathrm{cl}\,(K_1 \cap D_2)) < \epsilon$ also. But $w^* - \mathrm{cl}\,(K_1 \cap D_2) \supset (K \cap D_2)$ by w^*-density of the set $K \cap D_2$. Thus, $\operatorname{diam}(K \cap D_2) < \epsilon$. So we have proved that every w^*-compact subset of the dual space E^* is w^*-huskable. By the result of Kenderov [14] the space E^* possesses the RN-property. But the space E is separable, therefore by a result of Stegall [19] the space E^* is separable too. The proof is completed. ∎

2. Injections and c_o-subspaces

Let $T\colon E \to X$ be an injection of a Banach space E into a topological vector space X. We will say [8] that a subset $C \subset E$ is of the super-first category if and only if it can be covered by a countable union of X-closed and E-nowhere dense sets. We will use the following theorem [8].

THEOREM 4: *Let a Banach space E allow an injection $T\colon E \to X$ in some Hausdorff topological vector space X such that there exists a closed bounded solid (i.e. with non-empty interior) subset $A \subset E$ with the boundary ∂A of super-first category. Then $E \supset c_o$. Conversely, if a separable Banach space E contains a c_o-subspace, then there exist an injection $T\colon E \to X$ (into some Banach space X) even with T^{-1} from the first Baire class and an equivalent norm $|||.|||$ on the space E such that the unit sphere $S(E, |||.|||)$ (i.e. the boundary of the unit ball) is of super-first category.*

The following theorem is based on Theorem 4.

THEOREM 5: *Let $T\colon E \to X$ be an injection of a Banach space E into a Hausdorff topological vector space X. Let there exist a non-empty open bounded subset $G \subset E$ and a subset $C \subset G$ (possibly empty) of the super-first category such that the set $G \setminus C$ is a G_δ-set in the X-topology in the closure $E - \mathrm{cl}\, G$. Then $E \supset c_o$.*

Proof: Denote $A = E - \mathrm{cl}\, G$. By the conditions of the theorem, $G \setminus C = \bigcap G_n$ where each subset $G_n \subset A$ is X-open in A. Hence $A \setminus (G \setminus C) = (A \setminus G) \bigcup C = \cup V_n$, where each subset $V_n \subset A$ is X-closed in A. Since G is an open subset of the space E, it follows that $\partial A = A \setminus G$ and therefore $\partial A \subset \bigcup V_n$. It is clear that each subset V_n is X-closed in the set A and nowhere dense (V_n is norm-closed and $V_n \subset \partial A \cup C$). To complete the proof it remains to apply Theorem 4. ∎

Remark 4: If E is separable and $U(E)$ is closed in the X-topology then the condition: "$(G \setminus C)$ is a G_δ-set in X-topology in the set $E - \mathrm{cl}\, G$" is equivalent to the following one: "$(G \setminus C)$ is a G_δ-set in X-topology in the whole space E".

Before stating the corollary we introduce some notation. For subsets B and C of a Banach space E we will denote by

$$\delta(B, C) = \sup\{d(x, C)\colon x \in B\}$$

the deviation of the set B from the set C.

COROLLARY 2: *Let $T\colon E \to X$ be an injection of a Banach space E into a Hausdorff topological vector space X. Suppose that there exists a closed bounded* *subset $A \subset E$ possessing the property: $\partial A \subset \bigcup V_n$, $A \setminus \bigcup V_n \neq \emptyset$ where each subset V_n is an X-closed subset of the set A and $\lim \delta(V_n, \partial A) = 0$. Then $E \supset c_o$.*

Proof: Denote $G = A \setminus \bigcup V_n$. It is clear that the subset $G \subset A$ is a G_δ-set in X-topology on A. We will show that G is an open subset of the space E. Let $x \in G$. It is evident that $x \in (\mathrm{int} A)$ ($\mathrm{int} A$ is the interior of the set A in E-topology) and therefore there exists a number $r > 0$ such that $x + 2rU(E) \subset (\mathrm{int} A)$. Since for every $y + rU(E), d(y, \partial A) \geq r$ and $\lim \delta(V_n, \partial A) = 0$, it follows that there exists an integer m such that for every $n > m$, $V_n \cap (x + rU(E)) = \emptyset$. Denoting $\delta = \min\{1/2d(x, V_n), 1/2r: 1 \leq n \leq m\}$ we get $x + \delta U(E) \subset G$. Application of Theorem 5 ($C = \emptyset$) completes the proof. ∎

In connection with Theorem 5 it is interesting to consider the class $\mathcal{A}$ of all separable Banach spaces possessing the property: $E \in \mathcal{A}$ if and only if there exists an open bounded subset $G \subset E$ which is a G_δ-set in the space E in the weak topology.

PROPOSITION 2: *A separable Banach space E belongs to the class $\mathcal{A}$ if and only if there exists an open bounded subset $G \subset E$ which is a G_δ-set in the set $\|.\| - \mathrm{cl}\, G$ in w-topology.*

Proof: The proof follows from the observation: $E \setminus (\|.\| - \mathrm{cl}\, G)$ is an open subset of the separable Banach space E and hence it can be covered by a countable union of closed (both in norm and weak topologies) balls. ∎

Since the property $E \in \mathcal{A}$ is a hereditary one for subspaces of the space E, the next corollary follows from Theorem 5 ($C = \emptyset$).

COROLLARY 3: *Each Banach space from the class $\mathcal{A}$ contains an isomorphic copy of the space c_o hereditarily.*

Let us note that the class $\mathcal{A}$ is polar (but not opposite) to the class of Polish spaces [2, 18].

Remember that a Banach space is called polyhedral [15] if and only if the unit ball of every finite-dimensional subspace is a polyhedron.

PROPOSITION 3: *Every separable polyhedral Banach space belongs to the class $\mathcal{A}$.*

Proof: According to [3] a separable polyhedral Banach space E possesses a countable boundary, i.e. there exists a sequence $\{f_i\} \subset S(E^*)$ of linear functionals such that, for every $x \in E$, there exists functional f_j for which $f_j(x) = \|x\|$.

Let $G = (\text{int}U(E))$; then $G = \cap\{x \in E\colon f_i(x) < 1\}$ and therefore G is a G_δ-set in the space E in the w-topology. ∎

We conclude this section by the following

Problem: Is the property $E \in \mathcal{A}$ inherited by quotient spaces of the space E? What about polyhedral space E?

3. Resume

Theorem 7 is a consequence of the results of previous sections (Theorem 2, Theorem 4, Theorem 5 ($C = \emptyset$)) and the following result of the author [5].

THEOREM 6: *Let a separable Banach space E contain a BCBS. Then there exists a non-isomorphic semi-embedding $T\colon E \to X$ into some Banach space X.*

THEOREM 7: *Let E be a separable Banach space . Then the following assertions are equivalent:*

(1) $E \in \mathcal{K}$

(2) *There exist an injection $T\colon E \to X$ (into some Banach space X) and a non-empty bounded open subset $A \subset E$ such that either the image TA is of the type $G_{\delta\sigma}$ in the space X or the image TA is of the type G_δ in the image TE.*

(3) *There exist an injection $T\colon E \to Y$ (into some Banach space Y) and a non-empty bounded open subset $B \subset E$ such that either the image TB is of the type F_σ in the space Y or the image $T(cB)$ is of the type F_σ in the image TE.*

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